

# Fast graphs for the random walker

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## Abstract

Consider the time  $T_{oz}$  when the random walk on a weighted graph started at the vertex  $o$  first hits the vertex set  $z$ . We present lower bounds for  $T_{oz}$  in terms of the volume of  $z$  and the graph distance between  $o$  and  $z$ . The bounds are for expected value and large deviations, and are asymptotically sharp. We deduce rate of escape results for random walks on infinite graphs of exponential or polynomial growth, and resolve a conjecture of Benamini and Peres.

## 1 Introduction

A weighted graph  $G = (V, w)$  is a set of vertices with a symmetric nonnegative function  $w$  on  $V \times V$ ; the edges of  $G$  are given by the support of  $w$ . The goal of this paper is to give a lower bound for the hitting times of sets for reversible Markov chains, that is, random walks on weighted graphs. At each step, the walk chooses a neighboring site at random with odds given by the edge weights, and then moves there. The weight  $w_x$  of a vertex  $x$  can be defined as the sum of the weights over all incident edges. Define the weight  $w_z$  of a vertex set  $z$  as the sum of the weights of the vertices in the set. Let  $T_{oz}$  denote the first time the walk, started at  $o$  at time 0, visits the vertex or vertex set  $z$ .

Consider the simple random walk on the nearest neighbor graph of the integers from  $o = 0$  to  $z = n$  and unit edge weights. As it is easily computed,

$$\mathbf{E}T_{oz} = n^2.$$

Could the walk be faster if we assigned different edge weights? Consider the biased simple random walk on the same graph with odds of going left and right given by 1 and  $g > 1$ . This can be realized as a random walk on a weighted graph with weight  $g^n$  on the  $n$ th edge. We have

$$\mathbf{E}T_{oz} \sim n(g+1)/(g-1).$$

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The price we had to pay for higher speed is a higher weight on vertex  $z$ : the parameter  $w_z/w_o$  is constant 1 for the unbiased walk and  $g^{n-1}$  for the biased walk. This raises the question whether walks with fixed parameter  $w_z/w_o$  can be faster. Perhaps surprisingly, the first example, unbiased simple random walk, is not the fastest. Theorem 6 below implies that

$$\inf_G \mathbf{E}T_{oz} \sim \frac{n^2}{\log n},$$

where the infimum is taken over all weighted graphs (with possibly more complicated structure) with vertices  $o, z$  at distance  $n$  and  $w_z/w_o = 1$ . In contrast, the second example is asymptotically the fastest: by Corollary 4, the infimum of  $\mathbf{E}T_{oz}$  taken over all weighted graphs with vertices  $o, z$  at distance  $n$  and  $w_z/w_o = g^{n-1}$  is asymptotic to  $n(g+1)/(g-1)$ .

Let  $r_{oz}$  denote the effective resistance between vertices or vertex sets  $o, z$  when the graph  $G$  is thought of as an electric network and conductances are given by the edge weights. We are ready to state the main theorem.

**Theorem 1** *Let  $o$  be a vertex and  $z$  be a vertex set in a weighted graph with  $\text{dist}(o, z) \geq n+1$  for some integer  $n \geq 0$ . Let  $m_g, I_g(a)$  denote the mean and the large deviation rate function of the hitting time  $T'_{01}$  for biased simple random walk on the integers with odds 1 and  $g$  of going left and right, respectively. Then*

$$\begin{aligned} \mathbf{E}T_{oz} &\geq m_g n + 1, \\ \mathbf{P}(T_{oz} \leq an + 1) &\leq e^{-I_g(a)n}. \end{aligned}$$

Here  $g$  may be taken to be either

- (a) the  $g > 1$  solution of  $(g-1)^2 g^{n-2} = 2w_z/w_o$ , or
- (b)  $(w_z r_{oz})^{1/n}$ .

The classical formulas for  $m_g$  and  $I(a)$  are

$$m_g = \mathbf{E}T'_{01} = (g+1)/(g-1), \tag{1}$$

$$e^{-I_g(a)} = \frac{g}{a+1} \left( \frac{g}{a^2-1} \right)^{\frac{a-1}{2}} \left( \frac{2a}{g+1} \right)^a. \tag{2}$$

The implicit formula for the bound in part (a) can be made explicit.

**Fact 2** *Theorem 1 still holds if  $g$  is replaced by a greater quantity. Set  $\alpha := n^2 w_z/w_o \vee e$ , then an upper bound for  $g$  in part (a) is given by*

$$g' := \left[ \frac{5\alpha}{(\log \alpha)^2} \right]^{\frac{1}{n-2}}. \tag{3}$$

In its applications Theorem 1 is related to the bound of Varopoulos and Carne (1985), and in some cases, as in the corollaries below, it yields sharper results. Part (b) is related to the classical expression for commute time (see formula (43)) in the sense that it ties hitting times and resistance. Also note that Theorem 1 concerns large deviations, and therefore the bounds are more precise than what follows from the Brownian (or Central Limit Theorem)

scaling limit; it compares random walks on graphs directly to biased simple random walk. A version of the expected value bound of part (b) was published in an earlier paper, Virág (2000). Lee (1994ab) has a solution for the optimization problem for expected value in the case of simple path graphs and its continuous analogue. Large deviation questions in random trees are studied by Dembo, Gantert, Peres and Zeitouni (2001).

The most important step in proving Theorem 1 is a comparison of Laplace transforms.

**Proposition 3** *Using the notation of Theorem 1 (a), (b), respectively, the Laplace transform of  $T'_{0n} + 1$  dominates the Laplace transform of  $T_{oz}$ , that is, for  $\lambda \geq 0$  we have*

$$\mathbf{E}e^{-\lambda(T'_{0n}+1)} \geq \mathbf{E}e^{-\lambda T_{oz}}.$$

It is possible to take limits of Theorem 1 in many directions of its two parameters. The following asymptotic result shows that in expected value and large deviations, the fastest graphs are the ones corresponding to the asymmetric simple random walks.

**Corollary 4 (Large deviations for walks in graphs)**

*Let  $C, g > 1$  and let  $1 < a < m_g$ . Then*

$$\begin{aligned} \inf_G \mathbf{E}T_{oz} &= m_g n + O(1), \\ \sup_G \mathbf{P}[T_{oz} \leq an] &= e^{-I_g(a)n + o(n)}, \end{aligned}$$

*where the inf and sup are over all weighted graphs  $G$  with vertex  $o$  and vertex set  $z$  satisfying  $n \leq \text{dist}(o, z)$  and  $w_z/w_o \leq Cg^d$ . The functions  $O(1)$  and  $o(n)$  depend on  $n, C, g, a$  only.*

Corollary 4 allows us to prove a conjecture of Benjamini and Peres (see Peres (1999)), originally stated for unweighted trees. For an infinite graph  $G$  with a fixed vertex  $o$ , denote  $w_n$  the total weight on edges at distance  $n$ . The **exponential upper growth** of  $G$  is defined as  $\limsup w_n^{1/n}$ . Let  $|v|$  denote the graph distance between vertices  $v$  and  $o$ .

**Corollary 5 (Exponential growth and lim sup speed)** *Let  $G$  be an infinite weighted graph with exponential upper growth  $g_0$ , and let  $g = g_0 \vee 1$ . Then the random walk  $\{X_k\}$  on the graph satisfies*

$$\limsup \frac{|X_k|}{k} \leq \frac{g-1}{g+1} \quad a.s.$$

Note that equality holds for biased simple random walks, and, perhaps surprisingly, even in certain *recurrent* graphs (Example 15).

In another scaling, we have

**Theorem 6 (Large deviations in graphs of polynomial growth)**

*Let  $0 < c, d$  and  $0 < \alpha < 2/(p+2)$ . Then*

$$\begin{aligned} \inf_G \mathbf{E}T_{oz} &\sim \frac{2n^2}{(p+2)\log n}, \\ \sup_G \mathbf{P}[T_{oz} < \alpha n^2 / \log n] &= n^{-(\alpha(p+2)-2)^2/(8\alpha) + o(1)}. \end{aligned}$$

*The inf and sup are taken over graphs  $G$  with vertices  $o, z$  satisfying  $n \leq \text{dist}(o, z)$  and  $w_z/w_o < cn^p$ . The function  $o(1)$  converges to 0 as  $n \rightarrow \infty$  and depends on  $n, \alpha, c, p$  only.*

Surprisingly, Theorem 1 can be used to get sharp results in this polynomial scaling, which is unlike the usual domain for large deviation type bounds. Theorem 6 implies a version of Khinchin's Law of the Iterated Logarithm for random walks on infinite graphs. We say an infinite graph  $G$  has **polynomial boundary growth** with power  $p$  if  $w_n \leq Cn^p$  for all  $n$  and fixed  $C$ .

**Corollary 7 (Law of the single logarithm for walks on graphs)**

*We have*

$$\sup_G \left( \limsup \frac{|X_k|}{\sqrt{k \log k}} \right) = \frac{\sqrt{p+2}}{2},$$

*where the supremum is taken over random walks  $\{X_k\}$  on weighted graphs  $G$  of polynomial boundary growth with power  $p$ . The  $\limsup$  is taken in the almost sure sense.*

from known examples and the bounds of Varopoulos and Carne (1985). Barlow and Perkins (1989) construct an unweighted subtree of  $\mathbf{Z}^2$  where the rate of escape is, up to a constant, the same as in Corollary 7.

In Sections 2, 3, 4, 5 and 6 we present a proof of Proposition 3. These sections contain the main ideas of the paper, which are outlined in Section 2. In Section 6 it is showed that (b) of Proposition 3 implies (b) of Theorem 1. A bit of extra work is needed to prove part (a) Theorem 1, and this is done in Section 7. Implications of Theorem 1 for graphs of exponential growth (Corollary 4 and Corollary 5) are discussed in Section 8. The polynomial case, including Theorem 6 and Corollary 7, is studied in Section 9.

## 2 Outline of the proof

The proof of Theorem 1 follows easily from the Laplace domination statement from Proposition 3. The first step in the proof of this proposition is to interpret the Laplace transform probabilistically. But first, let us make some conventions and introduce some notation.

For simplicity, the vertices in  $z$  may be identified as a single vertex (still denoted  $z$ ), as it will not change any of the quantities compared. A simple restriction argument also shows that it suffices to prove the claim for finite weighted graphs  $G$ . We assume further that all vertices in  $V \setminus \{z\}$  are accessible from  $o$  without passing through  $z$ .

Let us call the object of our study a **stopped random walk law, SRWL**, defined as a quadruple  $(K, o, V, z)$ , where  $V$  is a finite set of vertices,  $K$  is a reversible transition kernel on  $V$ ,  $o, z \in V$  are vertices at which the random walk will be started and stopped, respectively. Since we are only interested in the walk before it reaches the vertex  $z$ , let us introduce the notation  $K_z$  for the transition probabilities of the walk killed at  $z$ , that is  $K_z(x, y) := K(x, y)\mathbf{1}(x \neq z)$ .

The Laplace transform of  $T_{oz}$  has the following probabilistic interpretation. Let  $\beta \in (0, 1]$ , and consider the random walk which moves as the walk defined by  $K_z$ , but is killed before each step with probability  $1 - \beta$ . Denote its kernel  $K_\beta(x, y) := \beta K_z(x, y)$ . Let  $S_\beta$  denote the probability that this walk survives to hit  $z$ . Then

$$S_\beta = \sum_{k=0}^{\infty} \mathbf{P}(T_{oz} = k) \beta^k = \mathbf{E} \beta^{T_{oz}}.$$

This means that  $S_\beta$ , as a function of  $-\log \beta$ , is the Laplace transform of  $T_{oz}$ .

The main difficulty in the proof of the proposition is that one has to do optimization over a complicated geometric structure, a graph. There are, however, graphs for which the optimization is fairly simple, for example for the SRWL **supported on a path**. The definition is that the corresponding graph structure is a finite path with  $o, z$  at the two endpoints. Another example is that of a **dead-end RW law**, defined as a stopped random walk law for which the probability of getting to  $z$  is 0. For such walks the optimization is trivial, since the parameter  $S_\beta$  is identically 0.

Fortunately, every SRWL can be “decomposed” into such simple SRWLs in way that is convenient for our problem. First we define some parameters that are natural for such decomposition. Let  $R_\beta$  denote the number of times the it visits  $o$ . Let  $\Gamma_\beta = R_\beta w_z / w_o$ . The parameter  $w_z r_{oz}$  of part (a) will not enter directly into our analysis, but through the bound

$$\Gamma_\beta = R_\beta w_z / w_o \leq R_1 w_z / w_o = w_z r_{oz}. \quad (4)$$

The inequality here is trivial, where the second equality comes from the well-known connection between random walks and electric networks. Indeed, the probability that a random walk on a graph started at vertex  $o$  visits vertex  $z$  before returning to  $o$  is given by  $1/(w_o r_{oz})$ . The number of hits to  $o$  before hitting  $z$  is therefore a geometric random variable with success probability  $1/(w_o r_{oz})$ , so its expected value is  $R_1 = r_{oz} w_o$ , just what we needed.

In short, Proposition 3 amounts to a comparison of the parameter  $S_\beta$  with the parameter  $\Gamma_\beta / R_\beta$  (part (a)), and with the parameter  $\Gamma_\beta$  (part (b)). These parameters are natural because of

### Proposition 8 (Decomposition of SRWLs)

Let  $\mathcal{K}$  be a SRWL and let  $\beta \in (0, 1)$ . There exist SRWLs  $\{\mathcal{K}_i\}_{i \in \Pi \cup \{o\}}$  and a probability distribution  $\alpha$  on  $\Pi \cup \{o\}$  so that  $\{\mathcal{K}_\pi\}_{\pi \in \Pi}$  are supported on paths,  $\mathcal{K}_o$  is a dead-end RW law,

- the parameters  $\Gamma_\beta, S_\beta$ , and  $R_\beta$  of  $\mathcal{K}$  equal the convex combination with coefficients  $\alpha_i$  for the corresponding parameters of the  $\mathcal{K}_i$ , and
- $\text{dist}(o, z)$  in  $\mathcal{K}$  is not more than the corresponding distance in the  $\mathcal{K}_i$ .

This proposition essentially says that the optimization can be done on convex combinations of SRWLs on trivial graphs. As it turns out, even this case is not completely straightforward, especially for part (a). The analysis is presented in Sections 5, 6 and 7.

The proof of Proposition 8 depends on a duality between stopped random walk laws and certain loss flows presented in the next section.

## 3 Random walks and flows

Proposition 8 claims that all SRWLs can be replaced by convex combinations of basic SRWLs; this suggests a representation of SRWLs as an elements of a linear space. It will be a space of loss flows.

Let  $(K, o, V, z)$  be a SRWL, and let  $\beta \in (0, 1)$ . Consider the Green kernel  $\mathcal{G}_\beta(x, y)$  defined by  $K_\beta$ , which gives the expected number of times the walk started at  $x$  visits  $y$ :

$\mathcal{G}_\beta(x, y) := \sum_{n=0}^{\infty} K_\beta^n(x, y)$ . Define the function  $f : V \times V \rightarrow \mathbb{R}_{\geq 0}$

$$f(x, y) := \mathcal{G}_\beta(o, x)K_\beta(x, y), \quad (5)$$

in words, the expected number of steps the walk defined by  $K_\beta$  makes from  $x$  to  $y$ . This function encodes the original random walk in a nice way. For example, it is easy to see that reversibility of  $K$  is reflected by the fact that for each cycle  $\pi = (x_0, \dots, x_\ell = x_0)$  and its reversal  $\pi'$  the function  $f$  satisfies

$$f(\pi) = f(\pi'). \quad (6)$$

Here, and in the sequel, a **function from  $V \times V$  applied to a path** will mean the product of the values of the function over the edges of the path.

Also, by comparison of the expected number of steps entering and leaving a vertex  $x \in V$  the following **node law** holds:

$$\beta(f(V, x) + \mathbf{1}(o = x))\mathbf{1}(x \neq z) = f(x, V). \quad (7)$$

We refer to  $f$  as a “loss flow” because it satisfies Kirkhoff’s node law for flows except for the factor  $\beta < 1$ .

It is also possible to reconstruct the random walk from the loss flow. Given a vertex set  $V$ , vertices  $o, z$ , a real  $\beta \in (0, 1)$  and a nonnegative function  $f$  on  $V^2$  satisfying (6) and (7), define the transition kernel

$$K_z(x, y) := \frac{f(x, y)}{\beta(f(V, x) + \mathbf{1}(o = x))}.$$

This kernel corresponds to a SRWL  $(K, o, V, z)$  for which the flow defined by (5) is  $f$ .

The relevant parameters  $S_\beta, R_\beta, \Gamma_\beta$  can be expressed using the function  $f$ . Clearly:

$$S_\beta = \sum_{x \in V} f(x, z), \quad R_\beta = 1 + \sum_{x \in V} f(x, o). \quad (8)$$

The parameter  $\Gamma_\beta$  is somewhat harder to express. Notice that for  $x \in V \setminus \{z\}$ ,  $y \in V$  the definition (5) and the fact that  $K_\beta(x, y) = \beta w(x, y)/w_x$  implies

$$f(x, y) = \mathcal{G}_\beta(o, x)\beta w(x, y)/w_x,$$

and therefore for  $(y, x) \in \text{supp}(f)$

$$\theta(x, y) := \frac{f(x, y)}{f(y, x)} \quad (9)$$

satisfies

$$\theta(x, y) = \frac{\mathcal{G}_\beta(o, x)w_y}{\mathcal{G}_\beta(o, y)w_x}. \quad (10)$$

Since  $w(x, z) = w_x K(x, z)$ , and  $w_z = \sum_x w(x, z)$ , the parameter  $\Gamma_\beta = \mathcal{G}_\beta(o, o)w_z/w_o$  can be written as

$$\Gamma_\beta = \sum_x \mathcal{G}_\beta(o, o)w_x K(x, z)/w_o, \quad (11)$$

where the sum runs over all neighbors of  $z$ . For every such vertex  $x$  we pick a simple path  $\pi_x = (x_0 = o, \dots, x_{\ell(x)} = x)$  for which  $K(\pi_x) > 0$ . Repeated use of equation (10) then transforms (11) to an expression purely in terms of the flow  $f$ :

$$\Gamma_\beta = \sum_x \theta(\pi_x) \mathcal{G}_\beta(o, x) K(x, z) = \sum_x \theta(\pi_x) f(x, z) / \beta. \quad (12)$$

We have expressed the three important parameters as functions of the loss flow  $f$ .

## 4 Decomposition of flows

Consider a SRWL  $(K, o, V, z)$ , and let  $f$  be a flow defined by this SRWL (5). Fix  $\theta$  as in (9). Consider the set  $F$  of nonnegative functions  $f_*$  on  $V^2$  satisfying (7, 9) (with  $f$  replaced by  $f_*$  in both), and  $\text{supp}(f_*) \subset \text{supp}(f)$ . Note that (9) implies (6), so for each  $f_*$  it is possible to define a SRWL for which  $f_*$  is the corresponding loss flow. Since the parameters  $\Gamma_\beta, S_\beta, R_\beta$  (12, 8) are clearly linear on  $F$ , the following lemma will suffice for the proof of Proposition 8.

### Lemma 9 (Decomposition of loss flows)

*Let  $\Pi$  be the set of simple paths from  $o$  to  $z$ . There exists a probability distribution  $\alpha$  on  $\Pi \cup \{o\}$  so that*

$$f = \alpha_o f_o + \sum_{\pi \in \Pi} \alpha_\pi f_\pi,$$

*where  $f_\pi \in F$  is supported on  $\pi$  and  $f_o \in F$  is supported on  $(V \setminus \{z\})^2$ .*

For the proof of this lemma, it is useful to know which  $\pi \in \Pi$  supports elements of  $F$ .

**Lemma 10** *Let  $\pi = (x_0 = o, x_1, \dots, x_\ell = z)$  be a simple path. There exists  $f_\pi \in F$  supported on  $\pi$  if and only if  $\theta_i := \theta(x_i, x_{i-1}) < \beta$  for all  $1 \leq i \leq \ell$ .*

PROOF. There is a unique solution  $f_\pi$  supported on  $\pi$  for the equations (7) and (9). It can be obtained inductively; set  $\theta_0 := 0$ , then

$$\begin{aligned} f_\pi(x_{i-1}, x_i) &= \prod_{j=1}^i \frac{\beta - \theta_{j-1}}{1 - \beta \theta_j}, \\ f_\pi(x_i, x_{i-1}) &= \theta_i f_\pi(x_{i-1}, x_i). \end{aligned}$$

The solution is nonnegative (equivalently,  $f_\pi \in F$ ) if and only if  $\theta_i := \theta(x_i, x_{i-1}) < \beta$  for all  $1 \leq i \leq \ell$ .  $\square$

PROOF OF LEMMA 9.  $F$  is a closed, bounded, convex subset of a finite dimensional vector space, so it equals the closed convex hull of its extreme points (e.g. for a bit of overkill, by the Krein-Milman Theorem). Thus it suffices to prove that all extreme points of  $F$  are supported on  $(V \setminus \{z\})^2$  or on simple paths. Indeed, let  $f_*$  be extreme point. Consider the directed graph on  $V$  where  $(x, y)$  is an edge iff  $f_*(y, x) < \beta f_*(x, y)$  (equivalently, if  $\theta(y, x) < \beta$  and  $f_*(x, y) \neq 0$ ). Consider the set  $V'$  of vertices which are connected to  $z$  by a path directed towards  $z$  in this graph.

First suppose that  $o \notin V'$ . Summing the node law (7) over elements of  $V'$  yields

$$\beta f_*(V, V') - \beta f_*(V, z) = f_*(V', V). \quad (13)$$

The definition of  $V'$  implies that for  $(x, y) \in (V \setminus V') \times V'$  we have  $f_*(y, x) \geq \beta f_*(x, y)$ ; summation yields

$$f_*(V', V \setminus V') \geq \beta f_*(V \setminus V', V'). \quad (14)$$

Adding (14), the trivial inequality  $f_*(V', V') \geq \beta f_*(V', V')$ , and (13) yields  $0 \geq \beta f_*(V, z)$  and therefore  $\text{supp}(f_*) \subset (V \setminus \{z\})^2$ .

The second case is when  $o \in V'$ , so there is a directed path  $\pi$  satisfying the assumptions of Lemma 10 and that  $f_*(x, y) > 0$  if  $y$  follows  $x$  in  $\pi$ . Thus there exists an  $f_\pi \in F$  supported on  $\pi$ , and for a small  $\varepsilon > 0$ , the function  $(1 + \varepsilon)f_* - \varepsilon f_\pi$  is nonnegative hence an element of  $F$ . As  $f_*$  is an extreme point,  $f_* = f_\pi$ , and the proof is complete.  $\square$

## 5 An array encoding a random walk law

The goal of this section is to extract the information in the elementary SRWLs of the decomposition in Proposition 8 into an array of numbers.

First assume that the chain is supported on a simple path  $\pi = (x_0 = o, x_1, \dots, x_\ell = z)$ . Define the quantity

$$s(x, y) := \frac{\beta f(x, y) - f(y, x)}{f(x, y) - \beta f(y, x)}. \quad (15)$$

We will express the relevant parameters in terms of the  $s(x, y)$ . For  $1 \leq i \leq n$ , the definition of  $f$  implies that  $s(x_{i-1}, x_i) \in (0, \beta]$ , and the node law (7) applied to  $x_i$  implies that

$$\beta f(x_{i-1}, x_i) - f(x_i, x_{i-1}) = f(x_i, x_{i+1}) - \beta f(x_{i+1}, x_i).$$

This makes the following product telescope:

$$s(\pi) = \frac{\beta f(x_{\ell-1}, z) - f(z, x_{\ell-1})}{f(o, x_1) - \beta f(x_1, o)}. \quad (16)$$

Since  $f(z, x_{\ell-1}) = 0$ , we get  $s(x_{\ell-1}, z) = \beta$ . The node law (7) applied to  $o$  yields

$$\beta(f(x_1, o) + 1) = f(o, x_1), \quad (17)$$

and this implies that the denominator of the right hand side of (16) also equals  $\beta$ . So if  $\pi_{-z}$  denotes the path  $\pi$  with its last vertex removed, then

$$S_\beta = f(x_{\ell-1}, z) = \beta s(\pi_{-z}). \quad (18)$$

Note that the expression (17) equals  $\beta R_\beta$ , which, together with the definition of  $s(o, x_1)$  yields

$$R_\beta = \frac{1 - \beta s(o, x_1)}{1 - \beta^2}. \quad (19)$$



Finally, we substitute (18) to (12) to get

$$\Gamma_\beta = \theta(\pi_{-z})s(\pi_{-z})/\beta = h(s(\pi_{-z}))/\beta \quad (20)$$

where

$$h(s) := s \frac{1 - s\beta}{\beta - s} \quad (21)$$

so that  $h(s(x, y)) = s(x, y) \times \theta(x, y)$ . Now we turn to the general case.

**Proposition 11 (Array representation of SRWLs)**

*Consider a SRWL, and let  $\beta \in (0, 1)$ . There exists*

- a finite index set  $\Pi$ ,
- positive numbers  $\alpha_\pi$  for  $\pi \in \Pi$  with total sum at most 1,
- positive integers  $\ell_\pi$  for  $\pi \in \Pi$ , with  $\ell_\pi \geq \text{dist}(o, z)$  and
- $s_{\pi,i} \in (0, \beta)$  for  $\pi \in \Pi$ ,  $1 \leq i < \ell_\pi$

so that the parameters of the SRWL satisfy

$$S_\beta = \beta \sum_{\pi \in \Pi} \alpha_\pi \prod_{i=1}^{\ell_\pi-1} s_{\pi,i}, \quad (22)$$

$$\Gamma_\beta = \sum_{\pi \in \Pi} \alpha_\pi \prod_{i=1}^{\ell_\pi-1} h(s_{\pi,i}), \quad (23)$$

$$R_\beta \leq \frac{2}{1 - \beta^2} \left( 1 - \beta \sum_{\pi \in \Pi} \alpha_\pi s_{\pi,1} \right). \quad (24)$$

PROOF. We use the notation and the results of the decomposition in Proposition 8, so we can assume that our SRWL is a convex combination of SRWLs supported on simple paths and a dead-end RW law. For every simple path component  $\pi = (x_0^\pi, \dots, x_{\ell_\pi}^\pi)$ , consider the stopped random walk law there, and the flow defined in Section 3 for this walk. Let  $s_{\pi,i} := s(x_{i-1}^\pi, x_i^\pi)$ , as defined above (15).

The parameters  $S_\beta$  and  $\mathcal{G}_\beta$  equal zero for the dead-end random walk law component. Thus (22) and (23) follow from linearity and the simple path case (formulas (18, 19)).

The rest of the proof concerns the bound (24) for the parameter  $R_\beta$ ; it is relevant for part (a) Theorem 1, but not for part (b), and should be omitted at first reading. For the dead-end random walk law component, the  $R_\beta$  is bounded above by the expected lifetime of the walker  $(1 - \beta)^{-1}$  (in fact, this is sharp, achieved when the graph consists of the vertex  $o$  and a self-loop; if we outlaw self-loops, the sharp bound becomes  $(1 - \beta^2)^{-1}$ ). From this and the simple path case (19) we get

$$R_\beta \leq \frac{\alpha_o}{1 - \beta} + \sum_{\pi \in \Pi} \alpha_\pi \frac{1 - \beta s_{\pi,1}}{1 - \beta^2}.$$

Unfortunately, because of the possible self loops, this expression for  $R_\beta$  is messy, making the solution of the optimization problem messy, too. To avoid this, we sacrifice sharpness for simplicity, bounding the  $(1 - \beta)^{-1}$  and  $(1 - \beta^2)^{-1}$  terms by  $2(1 - \beta^2)^{-1}$ . This yields the bound (24).  $\square$

## 6 Laplace domination

In this section we complete the proof of Proposition 3 part (b) and Theorem 1 part (b) outlined in Section 2. We will use the notation and results introduced above.

PROOF OF PROPOSITION 3, PART (B). We have seen in Section 2 that it suffices to bound  $S_\beta$  in terms of  $\Gamma_\beta$ . Using the notation and results of Proposition 11, we can write

$$\begin{aligned}
\Gamma_\beta &= \sum_{\pi} \alpha_{\pi} \prod_{i=1}^{\ell_{\pi}-1} h(s_{\pi,i}) \\
&\geq \sum_{\pi} \alpha_{\pi} h \left[ \left( \prod_{i=1}^{\ell_{\pi}-1} s_{\pi,i} \right)^{\frac{1}{\ell_{\pi}-1}} \right]^{\ell_{\pi}-1} \\
&\geq \sum_{\pi} \alpha_{\pi} h \left[ \left( \prod_{i=1}^{\ell_{\pi}-1} s_{\pi,i} \right)^{1/n} \right]^n \\
&\geq h \left[ \left( \sum_{\pi} \alpha_{\pi} \prod_{i=1}^{\ell_{\pi}-1} s_{\pi,i} \right)^{1/n} \right]^n = h \left[ (S_\beta/\beta)^{1/n} \right]^n.
\end{aligned}$$

The first inequality follows from Jensen's inequality and the fact that  $y \mapsto \log(h(e^y))$  is convex for  $y \leq \log \beta$ . The second, from the fact that the function  $y \mapsto h(y^{1/n})^n$  is increasing in  $n$  for  $y \in [0, 1]$ , and that  $\ell_{\pi} \geq n+1$ . The third inequality follows from Jensen's inequality and the fact that  $y \mapsto h(y^{1/n})^n$  is convex in  $y$  for  $y > 0$ .

Solving the above inequality for  $S_\beta$ , and using the fact (4) that  $g = (w_z r_{oz})^{1/n} \geq \Gamma_\beta^{1/n}$  we get

$$S_\beta \leq \beta \left( \frac{g+1 - \sqrt{(g+1)^2 - 4\beta^2 g}}{2\beta} \right)^n = \mathbf{E} \beta^{T'_{0n}+1}. \quad (25)$$

Note that  $T'_{0n}$  is the sum of  $n$  independent copies of  $T'_{01}$ . Conditioning on the first step yields  $\mathbf{E} \beta^{T'_{01}} = \beta(g + (\mathbf{E} \beta^{T'_{01}})^2)/(g+1)$ , and solving this equation gives the equality in (25), a standard result. Thus the inequality in (25), in terms of  $-\log \beta$ , is a comparison of the Laplace transforms of  $T_{oz}$  and  $T'_{0n} + 1$ , as required.  $\square$

PROOF OF THEOREM 1 PART (B). The expected value inequality follows from differentiating the Laplace transforms at 0. For the large deviation inequality, note that

$$\mathbf{P}(T_{oz} - 1 \leq an) \leq e^{\lambda an} \mathbf{E} e^{-\lambda(T_{oz}-1)}$$

for every  $\lambda > 0$  by Markov's inequality. Replacing the Laplace transform on the right by that of  $T'_{0n}$  we get

$$\begin{aligned}
\mathbf{P}(T_{oz} - 1 \leq an) &\leq \inf_{\lambda > 0} \mathbf{E} e^{-\lambda(T'_{0n}-an)} \\
&= \left( \inf_{\lambda > 0} \mathbf{E} e^{-\lambda(T'_{01}-a)} \right)^n = e^{-I(a)n}.
\end{aligned}$$

For the last equality, we used the fact that the infimum over  $\lambda \in \mathbb{R}$  is achieved when  $\lambda > 0$ ; this can be checked by direct calculation.

Direct computation, or the law of large numbers, implies the expression (1) for  $m_g$ , and a standard computation using the Laplace transform of  $T'_{01}$  yields its large deviation rate function  $I_g$  given by (2).  $\square$

## 7 Proof of the main theorem, part (a)

In this section we prove Theorem 1, part (a). It suffices to prove Proposition 3, part (a), a comparison of Laplace transforms. Given that, the proof of Theorem 1, part (b) in the previous section also implies part (a). The optimization needed here is much more complicated; the most technical part is presented separately at the end of the section in Lemma 13.

PROOF OF PROPOSITION 3, PART (A). Using the notation and results of Proposition 11, we can write

$$\begin{aligned} \Gamma_\beta &= \sum_{\pi} \alpha_{\pi} \prod_{i=1}^{\ell_{\pi}-1} h(s_{\pi,i}) \\ &\geq \sum_{\pi} \alpha_{\pi} h(s_{\pi,1}) h \left[ \left( \prod_{i=2}^{\ell_{\pi}-1} s_{\pi,i} \right)^{\frac{1}{\ell_{\pi}-2}} \right]^{\ell_{\pi}-2} \\ &\geq \sum_{\pi} \alpha_{\pi} h(s_{\pi,1}) h(s_{\pi,*})^{n-1}, \end{aligned} \tag{26}$$

where

$$s_{\pi,*} := \left( \prod_{i=2}^{\ell_{\pi}-1} s_{\pi,i} \right)^{\frac{1}{n-1}}.$$

The first inequality follows from Jensen's inequality and the fact that  $y \mapsto \log(h(e^y))$  is convex for  $y \leq \log \beta$ . The second, from the fact that the function  $y \mapsto h(y^{1/n})^n$  is increasing in  $n$  for  $y \in [0, 1]$ , and that  $\ell_{\pi} \geq n + 1$ .

We will keep the parameter  $S_{\beta} = \beta \sum_{\pi \in \Pi} \alpha_{\pi} s_{\pi,1} s_{\pi,*}^{n-1}$  fixed and try to minimize the lower bound (given by (26) and (24) of Proposition 11)

$$\frac{w_z}{w_o} = \frac{\Gamma_{\beta}}{R_{\beta}} > \left( \frac{1 - \beta^2}{2} \right) \frac{\sum_{\pi} \alpha_{\pi} h(s_{\pi,1}) h(s_{\pi,*})^{n-1}}{1 - \beta \sum_{\pi \in \Pi} \alpha_{\pi} s_{\pi,1}} \tag{27}$$

as the parameters  $s_{\pi,1}$  and  $s_{\pi,*}$  range over the set  $[0, \beta)$ . We first claim that the infimum is achieved on this set. If  $s_{\pi,1}$  converges to  $\beta$ , then  $h(s_{\pi,1})$  will converge to  $\infty$ , so the lower bound in (27) can only converge to a small value if  $h(s_{\pi,*})$  converges to 0, in which case  $s_{\pi,1} = 0$  is a better choice. The same argument can be made with the roles of  $s_{\pi,1}$  and  $s_{\pi,*}$  reversed, so the infimum must indeed be achieved on this set.

Lemma 13 below, where the hard part of the optimization is done, implies that  $s_{\pi,1}$  (respectively,  $s_{\pi,*}$ ) have to be the same for every  $\pi$ , so we may drop the indices  $\pi$ . The lower bound (27) reduces to

$$\frac{w_z}{w_o} > \left( \frac{1 - \beta^2}{2} \right) \frac{\alpha h(s_1) h(s_*)^{n-1}}{1 - \beta \alpha s_1}, \tag{28}$$

and we have  $S_\beta = \beta \alpha s_1 s_*^{n-1}$ . Since  $h(s_1)/s_1$  is increasing in  $s_1$ , it is clear that increasing  $\alpha$  while keeping  $\alpha s_1$  fixed will not change  $S_\beta$  but will decrease the numerator on the right hand side of (28). Therefore the minimum is achieved when  $\alpha$  is maximal, so we may take  $\alpha = 1$ . After cancellations, the bound (28) reduces to

$$\frac{w_z}{w_o} > \left( \frac{1 - \beta^2}{2} \right) \frac{h(s_*)^{n-1}}{\beta/s_1 - 1}. \quad (29)$$

We are left to minimize this while keeping  $s_1 s_*^{n-1}$  fixed. The solution is

$$\begin{aligned} s_1 &= \frac{1 - \beta^2}{(s_* - \beta)^2 + 1 - \beta^2} s_*, \\ \frac{1}{\beta/s_1 - 1} &= \frac{1 - \beta^2}{(\beta/s_* - 1)(1 - \beta s_*)} = \frac{1 - \beta^2}{(\beta/s_* - 1)^2} h(s_*)^{-1}. \end{aligned} \quad (30)$$

Clearly,  $s_1 < s_*$ , so we have  $S_\beta < \beta s_*^n$ . If we set  $g = h(s_*)$  then this gives exactly the inequality in formula (25). Therefore, to conclude the proof it suffices to show that  $g$  is bounded above by the  $g_0 > 1$  solution of  $2w_z/w_o = (g_0 - 1)^2 g_0^{n-2}$ . Equivalently, it suffices to show that  $2w_z/w_o \geq (g - 1)^2 g^{n-2}$ . This follows if we combine the simple bound

$$\frac{1 - \beta^2}{\beta/s_* - 1} > h(s_*) - 1 = g - 1$$

with formulas (29) and (30). □

**Remark 12** At the price of complicated and long computations, this proof can be modified to get the exact graph that maximizes the chance of survival  $S_\beta$  with  $w_z/w_o$  fixed for a given parameter  $\beta$ . This graph depends on  $\beta$ , and is a simple path graph, except that in some cases a self-loop appears at the vertex  $o$ . There are three points at which we sacrificed sharpness for simplicity: the bound (24) in Proposition 11 (this essentially eliminated the need for self-loops at  $o$ ), the bound for  $s_1 < s_*$  in the proof above, and the last inequality of the proof, which eliminated the dependence on  $\beta$ . If we do not allow self-loops at  $o$ , expected hitting time (the  $\beta \rightarrow 1$  case) is minimized in the graphs of Example 17.

**PROOF OF FACT 2.** For  $g > 1$  the expression  $(g - 1)^2 g^{n-2}$  is increasing in  $g$ , and can be bounded below by  $(\log g)^2 g^{n-2}$ , so it suffices to prove that this expression is at least  $2w_z/w_o$ . We substitute (3):

$$(\log g)^2 g^{n-2} = \left( \frac{1}{n-2} \log \left[ \frac{5\alpha}{(\log \alpha)^2} \right] \right)^2 \frac{5\alpha}{(\log \alpha)^2},$$

and replace the last  $5\alpha$  by  $5n^2 w_z/w_o$  to get the lower bound

$$\frac{w_z}{w_o} \left( \frac{n}{n-2} \log \left[ \frac{5\alpha}{(\log \alpha)^2} \right] \right)^2 \frac{5}{(\log \alpha)^2}.$$

This can be bounded below by  $w_z/w_o$  times

$$5 \left[ \frac{\log \alpha + \log 5 - 2 \log \log \alpha}{\log \alpha} \right]^2,$$

which is easily checked to be at least 2. □

**Lemma 13** *Suppose that  $x_1, y_1, x_2, y_2$  achieve the minimum of the expression*

$$\alpha_1 h(x_1) h(y_1)^m + \alpha_2 h(x_2)^m h(y_2) \quad (31)$$

*subject to the constraints*

$$\alpha_1 x_1 y_1^m + \alpha_2 x_2 y_2^m = c_1, \quad (32)$$

$$\alpha_1 x_1 + \alpha_2 x_2 \leq c_2, \quad (33)$$

$$0 \leq x_1, x_2, y_1, y_2 < \beta,$$

where  $h$  is defined in (21) and  $\alpha_i, c_i$  are positive constants. Then  $x_1 = x_2$  and  $y_1 = y_2$ .

PROOF.

*Step 1.* The minimum can only occur for  $x_i \leq y_i$ . Otherwise, we may define new values  $x'_i = y'_i = (x_i y_i^m)^{1/(m+1)}$ , this will not violate the constraints, and will decrease (31) by the convexity of  $x \mapsto \log h(e^x)$ .

*Step 2.* Minimum must be achieved in the interior of  $[0, \beta]^4$ . By step 1 and symmetry, the only other case is  $x_1 = 0$ , for this we may assume  $y_1 = 0$  (it makes no difference). Let  $c'_2$  be the value of the left hand side in (33). It is straightforward to check that the unique solution of (32) and

$$\alpha_1 x_1 + \alpha_2 x_2 = c'_2 \quad (34)$$

for which  $y_1 = y_2$  and  $x_1 = x_2$  gives a smaller value for (31).

*Step 3.* Minimum is in fact achieved when equality holds in (33), but we do not need to show this. From now on we will only use that (31) is also minimal when (33) is replaced by the equality constraint (34), where  $c'_2$  is the actual value of the left hand side of (33).

*Step 4.* Since we excluded the case that the minimum occurs on the boundary, it can only occur where the 4 dimensional gradient of (31) is perpendicular to the 2 dimensional surface determined by (32, 34), or the derivative evaluated at two linearly independent vectors tangent to this surface is 0. This means that the Jacobian of the map given by the function (31) and the left hand sides of (32, 34) has a two-dimensional nullspace, so any  $3 \times 3$  submatrix must be singular. The Jacobian is a  $4 \times 3$  matrix; the first and third columns are computed as

$$\begin{bmatrix} \alpha_1 h'(x_1) h(y_1)^m \\ \alpha_1 y_1^m \\ \alpha_1 \end{bmatrix}, \quad \begin{bmatrix} \alpha_1 m h(y_1)^{m-1} h'(y_1) h(x_1) \\ \alpha_1 m x_1 y_1^{m-1} \\ 0 \end{bmatrix}.$$

We get the other two columns by replacing the index 1 by 2. Now we set  $r(x) := h(x)/x$  and divide the first two columns by  $\alpha_i$ , and the last ones by entries in the second row:

$$\begin{bmatrix} h'(x_1)h(y_1)^m & h'(x_2)h(y_2)^m & r(y_1)^{m-1}h'(y_1)r(x_1) & r(y_2)^{m-1}h'(y_2)r(x_2) \\ y_1^m & y_2^m & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

The determinant of the right  $3 \times 3$  submatrix has to be 0, and this happens if and only if  $f(x_1, y_1) = f(x_2, y_2)$  with

$$f(x, y) = r(y)^{m-1}h'(y)r(x).$$

The left  $3 \times 3$  submatrix simplifies if we divide the first row by  $f(x_i, y_i)$ :

$$\begin{bmatrix} y_1^m \frac{h'(x_1)r(y_1)^m}{h'(y_1)h(x_1)} & y_2^m \frac{h'(x_2)r(y_2)^m}{h'(y_2)h(x_2)} & 1 \\ y_1^m & y_2^m & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

After computing the determinant, we get that this matrix is singular iff  $g(x_1, y_1) = g(x_2, y_2)$  with

$$g(x, y) = y^m \left( 1 - \frac{h'(x)}{r(x)} \frac{r(y)}{h'(y)} \right).$$

To complete the proof, we have to show that the map  $(x, y) \mapsto (f, g)$  is injective. This follows from the fact that if  $x \leq y$ , then  $f$  is increasing in  $x$ ,  $y$  and  $g$  is decreasing in  $x$  and increasing in  $y$ . Consider  $x_1, x_2, y_1, y_2$ ; the two interesting cases are  $x_1 < x_2, y_1 < y_2$ , and  $x_1 < x_2, y_1 > y_2$ . In the first case  $f(x_1, y_1) < f(x_1, y_2) < f(x_2, y_2)$ , in the second,  $g(x_1, y_1) > g(x_2, y_1) > g(x_2, y_2)$ .  $\square$

## 8 Results for graphs of exponential growth

This section contains the proofs of Corollary 4 and Corollary 5. We then give a recurrent example in which the inequality of Corollary 5 is sharp.

Note that biased simple random walks achieve the bounds of Corollary 4 by the Law of Large Numbers and the Large Deviation Principle. Thus it suffices to prove the following Corollary to Theorem 1. Its claim is more precise than the lower bound of Corollary 4.

**Corollary 14** *Let  $C, g > 0$  and let  $1 < a_0 < (g+1)/(g-1)$ . There exists  $C_1, C_2$  so that*

$$\begin{aligned} \mathbf{E}T_{oz} &> n(g+1)/(g-1) - C_2, \\ \mathbf{P}[T_{oz} \leq an] &< C_1 e^{-I_g(a)n} \end{aligned}$$

*for any weighted graph with vertices  $o, z$  satisfying  $n \leq \text{dist}(o, z)$  and  $w_z/w_o \leq Cg^d$ , and for all  $a \in [a_0, (g+1)/(g-1)]$ .*

PROOF. Set

$$\begin{aligned} n' &:= n - 1, \\ a' &:= \frac{an - 1}{n - 1}, \\ g' &:= \left[ \frac{5(n - 1)^2 C g^n}{[\log((n - 1)^2 C g^n)]^2} \right]^{1/(n-3)}. \end{aligned}$$

Note that  $g' > g$  as well as  $a' > a$ , and  $a', g'$  are bounded by some constants  $a'_{max}$  and  $g'_{max}$  for all  $n$ . Also, for all large  $n$ , we have  $1 < a' < (g' + 1)/(g' - 1)$ . For these  $n$  we apply Theorem 1 and Fact 2 with parameters  $n', Cg^n, a'$  to get

$$\mathbf{P}[T_{oz} \leq an] \leq e^{-n'I_{g'}(a')}, \quad (35)$$

$$\mathbf{E}T_{oz} \geq (g' + 1)(g' - 1)(n - 1) + 1. \quad (36)$$

The definition of  $g'$  implies that  $g'^{n-3}/g^{n-3} = O(1)$ , and therefore  $g' - g = O(1/n)$ . This and (36) proves the proposed expected value bound. Clearly  $a' - a = O(1/n)$ . The function  $(g, a) \rightarrow I_g(a)$  and its derivative are continuous on the set  $[g, g'_{max}] \times [a_0, a'_{max}]$ , and hence bounded. Therefore

$$\begin{aligned} nI_g(a) - n'I_{g'}(a') &\leq n(I_g(a) - I_{g'}(a)) + c \\ &\leq nc_1((g' - g) + (a' - a)) + c \leq c_2. \end{aligned} \quad (37)$$

The boundedness of  $I_g(a)$  implies that we can ignore the first few  $n$  for the price of increasing the constant  $C_1$ . Thus (35) and (37) imply the proposed large deviation bound.  $\square$

We are ready to prove Corollary 5.

PROOF OF COROLLARY 5. Let  $g' > g$  be arbitrary, and let  $T_n, w_n$  denote the hitting time and the total weight of the set of edges at distance  $n$  from  $o$ , respectively. Then

$$\limsup |X_k|/k = \limsup n/T_n. \quad (38)$$

For all large  $n$  we have  $w_n/w_o < g'^n$ , so by Proposition 8 each event  $T_n \leq an$  has probability at most  $c_1 e^{-c_2 n}$ . By the Borel-Cantelli Lemma only finitely many of these events happen. Thus (38) is at most  $(g' - 1)/(g' + 1)$ , and since  $g' > g$  was arbitrary, the Corollary follows.  $\square$

**Example 15** Let  $g \geq 2$  be an integer, and consider the graph of the nonnegative integers with  $g$ -ary trees of depth  $d_i$  attached at vertex  $i$  for every  $i$ . If  $d_i$  increases fast enough, then by the time the walk started from 0 visits the leaves of the tree at  $d_i$ , its speed will be nearly as high as the speed of the walk on the  $g$ -ary tree, and the upper growth of this graph is just  $g$ . This gives an example of a recurrent graph for which equality is achieved in Corollary 5.

## 9 Hitting times in graphs of polynomial growth

In this section we prove Theorem 6. One direction of the inequalities are simple corollaries to Theorem 1; the other direction is provided by the “fast graphs” of Example 17 (expected value) and Example 18 (large deviations).

In the end of the section, we prove Corollary 7, a version of Khinchin’s Law of the Iterated Logarithm. Again, we prove two inequalities; the first is provided by a Corollary to Theorem 6, the second, by Example 21.

### Corollary 16

Let  $0 < c, d$  and  $0 < \alpha < 2/(p+2)$ . We have

$$\mathbf{E}T_{oz} > \frac{2n^2}{(p+2)\log n} - O(n), \quad (39)$$

$$\mathbf{P}[T_{oz} \leq \alpha n^2 / \log n] < n^{-(\alpha(p+2)-2)^2/(8\alpha)+o(1)} \quad (40)$$

for all graphs  $G$  with vertices  $o, z$  satisfying  $n \leq \text{dist}(o, z)$  and  $w_z/w_o \leq cn^p$ . The functions  $o(1)$ ,  $O(n)$  depend on  $n, \alpha, c, p$  only.

PROOF. We apply Theorem 1, part (a) to the graph in question. The parameters we use are  $n' = n - 1$ ,  $a_n$  which is the solution of  $a_n n' + 1 = \alpha n^2 / \log n$ , and  $g_n = n^{(p+2)/n}$ . This will cover the graphs in question, since

$$(g_n - 1)^2 g_n^{n'} \sim \left( \frac{1}{n} \log n^{p+2} \right)^2 n^{p+2} = (p+2)n^p \log n,$$

and this dominates  $2w_z/w_o = 2cn^p$  for large  $n$ . The theorem yields  $\mathbf{E}T_{oz} > (1 + 2/(g_n - 1))n'$ , and using the fact that  $1/(g_n - 1) = 1/\log g_n + O(1)$ , the first claim (39) follows.

Theorem 1 also yields the bound (2) on  $\mathbf{P}[T_{oz} \leq \alpha n^2 / \log n]$  which we rewrite as follows:

$$\left[ 1 - \frac{(g-1)^2}{(g+1)^2} \right]^{\frac{a}{2}} \cdot g^{1/2} \cdot \left[ 1 + \frac{1}{a^2 - 1} \right]^{\frac{a-1}{2}} \cdot \left[ 1 - \frac{1}{a+1} \right].$$

Substituting the parameters for our case and taking logarithms we get

$$\frac{-a_n(\log g_n)^2}{8} + \frac{\log g_n}{2} + \frac{a_n}{2a_n^2} - \frac{1}{a_n} + o(n^{-1} \log n).$$

Multiplying by  $n$  and substituting the formulas for  $a_n$  and  $\log g_n$  we get

$$\log n \left( \frac{-\alpha(p+2)^2}{8} + \frac{p+2}{2} - \frac{1}{2\alpha} + o(1) \right).$$

Exponentiation yields the bound (40). □



**Example 17** We now show a family of fast simple path graphs of polynomial growth. Let  $g > 1$ , and consider the simple path graph with vertices denoted  $o = 0, 1, \dots, n = z$ , edges  $e_i = (i - 1, i)$  and edge weights

$$\begin{aligned} w(e_1) &= 1, \\ w(e_i) &= (g - 1)g^{i-2} \quad \text{for } 2 \leq i \leq n - 1, \\ w(e_n) &= (g - 1)^2 g^{n-3}. \end{aligned}$$

Heuristically, the walker has a positive drift when it is away from the endpoints of the path; the price is large negative drifts at the two ends.

Consider the stopped random walk on this graph and the flow associated with parameter  $\beta = 1$  as defined in (5). This flow is uniquely determined by the flow property (7) and the requirement that  $f(i, i+1)/f(i, i-1) = w(i, i+1)/w(i, i-1)$ . These equations have solution:

$$\begin{aligned} f(1, 0) &= g/(g - 1)^2, \\ f(n, n - 1) &= 0, \\ f(i, i - 1) &= 1/(g - 1) \quad \text{for } 2 \leq i \leq n - 1, \\ f(i - 1, i) &= f(i, i - 1) + 1 \quad \text{for } 1 \leq i \leq n. \end{aligned}$$

And  $f(i, j) = 0$  elsewhere. Then clearly

$$\mathbf{E}T_{oz} = \sum_{i,j} f(i, j) = 2(n - 2)/(g - 1) + 2g/(g - 1)^2 + n.$$

Now consider the case when

$$\begin{aligned} g &= \left[ \frac{n^{p+2}}{(\log(n^{p+2}))^2} \right]^{1/n}, \quad \text{so that} \\ g - 1 &\sim \log g \sim \frac{\log(n^{p+2})}{n}. \end{aligned} \tag{41}$$

This example proves one direction of the expected value bound in Theorem 6, since

$$\begin{aligned} w_z/w_o &= (g - 1)^2 g^{n-3} \sim n^p, \\ \mathbf{E}T_{oz} &\sim \frac{2n^2}{(p + 2) \log n}. \end{aligned}$$

**Example 18** We now show that in the previous example the large deviation bounds of Proposition (16) are also achieved. It is perhaps surprising that the bounds are sharp even in this scaling. We want to estimate the probability that the hitting time is short by dividing the path into three segments. Let

$$m := \lfloor n/\log n \rfloor, \quad v := n - 1.$$

We expect the walk to spend most of its time between the vertices  $m$  and  $v$ . More precisely, let

$$\begin{aligned} t &:= \alpha n^2 / \log n, \\ t' &:= t/(\log n)^{1/2} = \alpha n^2 / (\log n)^{3/2}, \end{aligned}$$

so by the strong Markov property

$$\mathbf{P}[T_{oz} < t] \geq \mathbf{P}[T_{mv} < t - t'] \mathbf{P}[T_{om} + T_{vz} < t']. \quad (42)$$

The second factor can be bounded using the classical formula for commute time (see Chandra et al (1989)) and Markov's inequality. For any weighted graph and vertices  $o, z$ , if  $r_{oz}$  denotes effective resistance, then

$$\mathbf{E}[T_{oz} + T_{zo}] = w_V r_{oz}. \quad (43)$$

By the series rule

$$\begin{aligned} r_{om} &= \sum_{i=1}^m w(e_i)^{-1} = 1 + \frac{1 - (1/g)^{m-1}}{(g-1)(1-1/g)} \\ &\asymp (g-1)^{-2} \asymp n^2/(\log n)^2, \end{aligned} \quad (44)$$

the same way we get

$$r_{oz} \asymp n^2/(\log n)^2, \quad (45)$$

and  $r_{vz} \sim n^{-p}$ . Also, the total sum of edge weights satisfies

$$w_E = 1 + g^{n-3}(g-1)^2 + (g-1) \sum_{i=2}^{n-1} g^{i-2} \asymp n^p + g^{n-2} \asymp \frac{n^{p+2}}{(\log n)^2}.$$

For the edges  $E'$  on the path between vertices  $o, m$  we have

$$w_{E'} = 1 + (g-1) \sum_{i=2}^m g^{i-2} \asymp (g^{m-1} - 1) \asymp 1,$$

so we can apply (43) twice:

$$\mathbf{E}[T_{om} + T_{vz}] \leq 2w_{E'} r_{om} + 2w_E r_{vz} \asymp \frac{n^2}{(\log n)^2}.$$

Markov's inequality concludes the bound on the chance of the complement of the last event of (42):

$$\mathbf{P}[T_{om} + T_{vz} \geq t'] \leq \frac{\mathbf{E}[T_{om} + T_{vz}]}{t'} = O(\log n)^{1/2}.$$

It remains to bound the first factor:

$$\mathbf{P}[T_{mv} < t - t'] \geq \mathbf{P}[T_{mv} < t - t' \mid T_{mv} < T_{mo}] \mathbf{P}[T_{mv} < T_{mo}].$$

The second term here can be computed using resistances:

$$\mathbf{P}[T_{mv} < T_{mo}] = \frac{1}{1 + r_{mv}/r_{mo}} \geq \frac{1}{1 + r_{ov}/r_{mo}} \geq c > 0.$$

The constant lower bound follows from (44) and (45). To bound the first term, first note that the walk started at  $m$  and conditioned on the event  $T_{mv} < T_{mo}$  is a Doob transform of

the original walk, a reversible random walk in which the forward drift is bounded below by the forward drift in the original walk. Therefore, by stochastic domination,

$$\mathbf{P}[T_{mv} < t - t' \mid T_{mv} < T_{mo}] \geq \mathbf{P}[T'_{0,n-1-m} < t - t'],$$

where  $T'$  denotes hitting time for biased simple random walk  $\{X'_k\}$  on the integers with odds of going left and right equal  $1 : g$ . The second probability is bounded below by the probability of a smaller event, which in turn can be bounded using Lemma 19:

$$\mathbf{P}[X'_{[t-t'-1]} \geq n - 1 - m] > n^{-(\alpha(p+2)-2)^2/(8\alpha)+o(1)}.$$

All together, this example gives one direction in the large deviation bound of Theorem 6:

$$\mathbf{P}[T_{oz} < \alpha n^2 / \log n] > n^{-(\alpha(p+2)-2)^2/(8\alpha)+o(1)}.$$

We now turn to the proof of the simple lemma we used in the previous example. We were unable to locate a theorem in the literature that would imply this claim.

**Lemma 19** *Let  $p \geq 0$  and  $\{X_k\}$  be biased simple random walk on the integers with odds for going left and right given by 1 and  $g = g(p, n)$  defined in formula (41). Let  $\alpha < 2/(p+2)$  and let  $t = t(n) \sim \alpha n^2 / \log n$ . Then*

$$\mathbf{P}[X_t \geq n] > n^{-(\alpha(p+2)-2)^2/(8\alpha)+o(1)}.$$

PROOF. Without loss of generality we may assume that  $t$  is even. Let  $n_1 = n(1+1/\log n)$ . Then

$$\begin{aligned} \mathbf{P}[X_t \geq n] &\geq \mathbf{P}[X_t \in [n, n_1]] \\ &\geq \frac{n_1 - n - 2}{2} \inf_{\substack{m \\ \text{even} \\ n \leq m \leq n_1}} \mathbf{P}[X_t = m]. \end{aligned} \tag{46}$$

We now use the binomial formula to get that for  $t, m$  even

$$\mathbf{P}[X_t = m] = \binom{t}{(t+m)/2} \frac{g^{(t+m)/2}}{(1+g)^t}.$$

By Stirling's formula and the fact that  $t(n), m(n) \rightarrow \infty$  we get

$$\begin{aligned} 2^{-t} \binom{t}{(t+m)/2} &\sim \frac{1}{\sqrt{\pi}} \frac{t^{t+1/2}}{(t+m)^{(t+m+1)/2} (t-m)^{(t-m+1)/2}} \\ &\asymp t^{-1/2} \left[ 1 + \frac{m^2}{t^2 - m^2} \right]^{t/2} \left[ 1 - \frac{2m}{t+m} \right]^{m/2}. \end{aligned}$$

The remaining factor can be written as

$$2^t \frac{g^{(t+m)/2}}{(1+g)^t} = g^{m/2} \left[ 1 - \frac{(1-g)^2}{(1+g)^2} \right]^{t/2}.$$

Using the expression (41) for  $g$ , the fact that  $g(n) \rightarrow 1$  and that  $m(n)/t(n) \rightarrow 0$  we get that

$$\begin{aligned} \log \mathbf{P}[X_t = m] &\sim -\frac{\log t}{2} + \frac{m^2}{t^2} \cdot \frac{t}{2} - \frac{2m}{t} \cdot \frac{m}{2} + \frac{m}{2} \log g - \frac{(\log g)^2}{4} \cdot \frac{t}{2} \\ &\sim \left(-1 + 1/(2\alpha) - 1/\alpha + (p+2)/2 - (p+2)^2\alpha/8\right) \log n \\ &= \left(-1 - (\alpha(p+2) - 2)^2/(8\alpha)\right) \log n. \end{aligned}$$

The convergence is uniform over all  $m \in [n, n_1]$ . This and (46) imply the claim of the lemma.  $\square$

We now turn to the proof of the graph version of Khinchin's Law of the Iterated Logarithm. The upper bound is a Corollary to Theorem 6.

**Corollary 20 (Law of the single logarithm, upper bound)**

*For random walks  $\{X_k\}$  on weighted graphs with polynomial boundary growth with power  $p$  we have*

$$\limsup \frac{|X_k|}{\sqrt{k \log k}} \leq \frac{\sqrt{p+2}}{2} \quad a.s. \quad (47)$$

PROOF. Let  $a = 2/(p+2)$ , let  $a'' < a' < a$ , and let

$$f(t) = \sqrt{(t \log t)/(2a'')}. \quad (48)$$

Let  $m > 1$  an integer, for every  $k$ , let  $\ell_k$  denote the distance of the farthest vertex visited up to time  $k$ , and let  $\ell'_k$  be the greatest integer so that  $\ell'^m_k < \ell_k$ . Then

$$\frac{|X_k|}{f(k)} \leq \frac{\ell_k}{f(T_{\ell_k})} \leq \frac{(\ell'_k + 1)^m}{f(T_{\ell'^m_k})} = \frac{(\ell'_k + 1)^m}{\ell'^m_k} \frac{\ell'^m_k}{f(T_{\ell'^m_k})}.$$

Taking lim sup we get

$$\limsup_{k \rightarrow \infty} \frac{|X_k|}{f(k)} \leq \lim_{k \rightarrow \infty} \frac{(\ell'_k + 1)^m}{\ell'^m_k} \limsup_{k \rightarrow \infty} \frac{\ell'^m_k}{f(T_{\ell'^m_k})} = \limsup_{\ell \rightarrow \infty} \frac{\ell^m}{f(T_{\ell^m})}. \quad (49)$$

We are taking  $m$ th powers to make a sequence of probabilities summable. Now consider the function

$$g(\ell) = a'\ell^2 / \log \ell, \quad (50)$$

an upper bound for  $f^{-1}$ , so that  $t < g(f(t))$  for all large  $t$ . For all large  $\ell$   $g(\ell)$  is increasing, and we have

$$\begin{aligned} \mathbf{P}[f(T_\ell) < \ell] &= \mathbf{P}[g(f(T_\ell)) < g(\ell)] \\ &\leq \mathbf{P}[T_\ell < g(\ell)] \\ &\leq \ell^{-(a'(p-2)-2)^2/(8a')+o(1)}. \end{aligned}$$

The last inequality follows from Theorem 6. For large  $\ell$  the right hand side is bounded above by  $\ell^{-c}$  for some  $c > 0$ , so it is summable over the subsequence of  $m$ -powers if  $cm > 1$ . Therefore by the Borel-Cantelli lemma  $f(T_{\ell^m}) \geq \ell^m$  eventually a.s., so the right hand side of (49) is at most 1. Since  $a'' < a$  was arbitrary, the corollary follows.  $\square$

**Example 21** Using Example 17 it is easy to construct an example for the sharpness of Corollary 20, and thus prove Corollary 7. Let  $x_i$  be a sequence where  $x_i - x_{i-1}$  is positive and rapidly increasing. Consider the sequence of simple path graphs  $G_i$  of length  $n_i = x_i - x_{i-1}$  and of polynomial growth  $w_z/w_o = n_i^p$  constructed in Example 17. We concatenate them in increasing order to get an infinite simple path graph. By picking  $x_i - x_{i-1}$  to be rapidly increasing, it can be achieved that the dominant term in the expected hitting time  $\mathbf{ET}_{0,x_i}$  will be the expected hitting time  $\mathbf{ET}_{o_z}$  in the graph  $G_i$ . This means that if we set  $a = 2/(p+2)$  the walk in the concatenated graph has

$$\mathbf{ET}_{0,x_i} \sim a \frac{x_i^2}{\log x_i}.$$

Let  $a'' > a' > a$ . With  $g$  as in (50), by Markov's inequality, for all large  $i$  and all  $y < x_i$  we have

$$\mathbf{P}[T_{y,x_i} \leq g(x_i)] \geq 1 - a/a'.$$

The function  $f$  (48) is a lower bound for the inverse of  $g$  in the sense that  $f(g(\ell)) < \ell$  for all large  $\ell$ . Therefore for large  $i$

$$\mathbf{P}[f(T_{y,x_i}) \leq x_i] \geq 1 - a/a'.$$

The following implications are simple:

$$\limsup_{i \rightarrow \infty} \left\{ \frac{x_i}{f(T_{x_i})} \geq 1 \right\} \Rightarrow \limsup_{k \rightarrow \infty} \left\{ \frac{X_k}{f(k)} \geq 1 \right\} \Rightarrow \left\{ \limsup_{k \rightarrow \infty} \frac{X_k}{f(k)} \geq 1 \right\}.$$

The first event has probability at least  $1 - a'/a$  no matter which vertex the walk is started at. Let  $A$  denote the last event; we then have  $\mathbf{P}A \geq 1 - a'/a$  even if we start the walk at a different time (as opposed to time 0). Thus by Lévy's 0-1 law

$$1 - a'/a \leq \mathbf{P}[A|X_0, \dots, X_k] \rightarrow \mathbf{1}_A \quad a.s.$$

Thus  $\mathbf{P}A = 1$ , and since  $a'' > a$  was arbitrary, the lower bound in Corollary 7 follows.

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